Perturbative Noncommutative Quantum Gravity

J. W. Moffat

Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

Abstract

We study perturbative noncommutative quantum gravity by expanding the gravitational field about a fixed classical background. A calculation of the one loop gravitational self-energy graph reveals that only the non-planar graviton loops are damped by oscillating internal momentum dependent factors. The noncommutative quantum gravity perturbation theory is not renormalizable beyond one loop for matter-free gravity and all loops for matter interactions. Comments are made about the nonlocal gravitational interactions produced by the noncommutative spacetime geometry.

1 Introduction

There has been renewed attention paid to noncommutative field theories in view of their appearance in string theory and D-brane theories [1, 2, 3]. The noncommutative geometry is characterized by d noncommuting self-adjoint operators $\hat{x}^\mu$ in a Hilbert space $\mathcal{H}$ satisfying

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a non-degenerate $d \times d$ antisymmetric matrix. Given an operator $\hat{\phi}$ associated with a function $\phi(x)$ on the commutative space, the operator $\hat{\phi}$ acts in the Hilbert space $\mathcal{H}$ according to the Weyl correspondence

$$\hat{\phi}(\hat{x}) = \frac{1}{(2\pi)^d}\int d^dxd^dk\exp[ik_\mu(\hat{x}^\mu - x^\mu)]\phi(x). \tag{2}$$

The function $\phi(x)$ can be derived from

$$\phi(x) = \frac{1}{(2\pi)^{d/2}}\int d^dk\exp(ik_\mu x^\mu)\mathrm{tr}[\hat{\phi}(\hat{x})\exp(-ik_\mu\hat{x}^\mu)], \tag{3}$$

where the trace operation tr is over the Hilbert space $\mathcal{H}$.

The product of two operators $\hat{\phi}_1$ and $\hat{\phi}_2$ has a corresponding Moyal $\star$-product

$$(\hat{\phi}_1 \star \hat{\phi}_2)(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\zeta^\nu}\right)\phi_1(x+\xi)\phi_2(x+\zeta)|_{\xi=\zeta=0}. \tag{4}$$

The problem with perturbative quantum gravity based on a commutative spacetime and a local field theory formalism is that the theory is not renormalizable [4, 5]. Due to the Gauss-Bonnet theorem, it can be shown that the one

loop graviton calculation is renormalizable but two loop is not [6]. Moreover, gravity-matter interactions are not renormalizable at any loop order.

Recently, the consequences for gravitation theory of using the Moyal $\star$-product to define a general quantum gravity theory on a noncommutative spacetime were analyzed, and it was found that a complex symmetric metric (non-Hermitian) theory could possibly provide a consistent underlying geometry on a complex coordinate manifold [7]. In the following, we shall investigate the consequences for perturbative quantum gravity, when the gravitational action is given on a noncommutative spacetime geometry. We expand the metric about a flat Minkowski spacetime by taking the usual Einstein-Hilbert action, whose fields are functions on ordinary noncommutative spacetime, except that the products of field quantities are formed by using the Moyal $\star$-product rule.

We treat $\theta^{\mu\nu}$ as an antisymmetric two-tensor with $\theta_{\mu\nu}\theta^{\mu\nu} < 0$. The latter requirement guarantees that the quantum theory is bounded from below in its lowest critical value. We shall restrict ourselves to the case $\theta^{ij} \neq 0$ and $\theta^{0i} = 0$, because the $\theta^{0i} \neq 0$ case has peculiar causal and quantum properties. The problem with time-space noncommutativity is due to the fact that one cannot define the perturbative Hilbert space, because the conjugate momentum of a field is ill-defined, and we cannot know what the correct measure for the path integral should be. The perturbative results for the gravitational field are derived by using the modified graviton Feynman rules obtained by Filk [8], and other authors [9, 10, 11, 12, 13, 14, 15, 16, 17].

The first order, one loop graviton self-energy is calculated in Sect. 3, using a noncommutative action and functional generator $Z[j_{\mu\nu}]$ [18, 19, 20, 4, 21, 22, 23, 24, 25, 26]. We find that the planar one loop graviton graph and vacuum polarization are essentially the same as for the commutative perturbative result, while the non-planar graviton loop graph is damped due to the oscillatory behavior of the noncommutative phase factor in the Feynman integrand. Thus, the overall noncommutative perturbative theory remains unrenormalizable and divergent. In Sect. 4, we consider the nonlocal nature of the gravitional interactions in the noncommutative theory and in Sect 5, we end with a summary of the results.

2 The Noncommutative Gravity Action

We define the noncommutative gravitational action as

$$S_{\rm grav} = -\frac{2}{\kappa^2}\int d^4x(\sqrt{-g}\star R + 2\sqrt{-g}\lambda), \tag{5}$$

where we use the notation: $\mu,\nu = 0,1,2,3$, $g = \det(g_{\mu\nu})$, the metric signature of Minkowski spacetime is $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$, $R = g^{\mu\nu}\star R_{\mu\nu}$ denotes the scalar curvature, λ is the cosmological constant and $\kappa^2 = 32\pi G$ with $c = 1$. The Riemann tensor is defined such that

$$R^\lambda{}_{\mu\nu\rho} = \partial_\rho\Gamma_{\mu\nu}{}^\lambda - \partial_\nu\Gamma_{\mu\rho}{}^\lambda + \Gamma_{\mu\nu}{}^\alpha\star\Gamma_{\rho\alpha}{}^\lambda - \Gamma_{\mu\rho}{}^\alpha\star\Gamma_{\nu\alpha}{}^\lambda. \tag{6}$$

We shall expand the gravity sector about flat Minkowski spacetime. In fact, we can expand around any fixed, classical metric background [4]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{7}$$

where $\bar{g}_{\mu\nu}$ is any smooth background metric field. For the sake of simplicity, we shall only consider expansions about flat, four-dimensional spacetime and we choose $\lambda = 0$. Since the gravitational field is weak up to the Planck energy scale, this expansion is considered justified up to the latter scale; even at the standard model energy scale $E_{\rm SM} \sim 10^2$ GeV, we have $\kappa E_{\rm SM} \sim 10^{-16}$. At these energy scales the curvature of spacetime is very small.

Let us define $\mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$. It can be shown that $\sqrt{-g} = \sqrt{-\mathbf{g}}$, where $\mathbf{g} = \det(\mathbf{g}^{\mu\nu})$ and $\partial_\rho \mathbf{g} = \mathbf{g}_{\alpha\beta}\partial_\rho \mathbf{g}^{\alpha\beta}\mathbf{g}$. We expand the local interpolating graviton field $\mathbf{g}^{\mu\nu}$ as

$$\mathbf{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\gamma^{\mu\nu} + O(\kappa^2). \tag{8}$$

Then, for the noncommutative spacetime

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} - \kappa\gamma_{\mu\nu} + \kappa^2 {\gamma_\mu}^\alpha \star \gamma_{\alpha\nu} - \kappa^3 {\gamma_\mu}^\alpha \star \gamma_{\alpha\beta} \star {\gamma_\nu}^\beta + O(\kappa^4). \tag{9}$$

We can now write the noncommutative gravitational action $S_{\rm grav}$ in the form:

$$S_{\rm grav} = \frac{1}{2\kappa^2}\int d^4x[(\mathbf{g}^{\rho\sigma} \star \mathbf{g}_{\lambda\mu} \star \mathbf{g}_{\kappa\nu}$$

$$-\frac{1}{2}\mathbf{g}^{\rho\sigma} \star \mathbf{g}_{\mu\kappa} \star \mathbf{g}_{\lambda\nu} - 2\delta^\sigma_\kappa \delta^\rho_\lambda \mathbf{g}_{\mu\nu}) \star \partial_\rho \mathbf{g}^{\mu\kappa} \star \partial_\sigma \mathbf{g}^{\lambda\nu}]. \tag{10}$$

Since the free part of the action is the same as the commutative case, the vacuum state and the quantization procedure are the same as in the commutative case. Only the interaction part is linked to the noncommutative case. In particular, the measure in the functional integral formalism is the same as the commutative theory, for in momentum space the additional phase factor disappears when we impose the normalization condition for the partition function.

Let us consider the noncommutative generating functional [20, 21, 23]

$$Z[j_{\mu\nu}] = \int d[\mathbf{g}^{\mu\nu}]\Delta[\mathbf{g}^{\mu\nu}] \exp i\biggl[S_{\rm grav} + \frac{1}{\kappa}\int d^4x \mathbf{g}^{\mu\nu} j_{\mu\nu}$$

$$-\frac{1}{\kappa^2\alpha}\int d^4x \partial_\mu \mathbf{g}^{\mu\nu} \star \partial_\alpha \mathbf{g}^{\alpha\beta}\eta_{\nu\beta}\biggr], \tag{11}$$

where $(\partial_\mu \mathbf{g}^{\mu\nu} \star \partial_\alpha \mathbf{g}^{\alpha\beta}\eta_{\nu\beta})/\kappa^2\alpha$ is the gauge fixing term. Here, Δ can be interpreted in terms of fictitious particles and is given by

$$\Delta[\mathbf{g}^{\mu\nu}]^{-1} = \int d[\xi_\lambda]d[\eta_\nu] \exp i\biggl\{\int d^4x \eta^\nu \star [\eta_{\nu\lambda}\partial^\sigma\partial_\sigma - \kappa(\partial_\lambda\partial^\mu\gamma_{\mu\nu} - \gamma_{\mu\rho}$$

$$\eta_{\nu\lambda}\partial^\mu\partial^\rho - \partial^\mu\gamma_{\mu\rho}\eta_{\nu\lambda}\partial^\rho + \partial^\mu\gamma_{\mu\nu}\partial_\lambda)] \star \xi^\lambda\biggr\}, \tag{12}$$

where ξ^λ and η^λ are the fictitious ghost particle fields.

The gravitational action is expanded as

$$S_{\rm grav} = S^{(0)}_{\rm grav} + \kappa S^{(1)}_{\rm grav} + \kappa^2 S^{(2)}_{\rm grav} + ... \tag{13}$$

We obtain

$$S^{(0)}_{\rm grav} = \frac{1}{2}\int d^4x[\partial_\sigma\gamma_{\lambda\rho}\partial^\sigma\gamma^{\lambda\rho} - \partial_\lambda\gamma^{\rho\kappa}\partial_\kappa\gamma^\lambda_\rho - \frac{1}{4}\partial_\rho\partial^\rho\gamma - \frac{1}{\alpha}\partial_\rho\gamma^\rho_\lambda\partial_\kappa\gamma^{\kappa\lambda}], \tag{14}$$

$$S^{(1)}_{\rm grav} = \int d^4x\frac{1}{4}[(-4\gamma_{\lambda\mu}\star\partial^\rho\gamma^{\mu\kappa}\star\partial_\rho\gamma_\kappa{}^\lambda + 2\gamma_{\mu\kappa}\star\partial^\rho\gamma^{\mu\kappa}\star\partial_\rho\gamma$$
$$+2\gamma^{\rho\sigma}\star\partial_\rho\gamma_{\lambda\nu}\star\partial_\sigma\gamma^{\lambda\nu} - \gamma^{\rho\sigma}\star\partial_\rho\gamma\star\partial_\sigma\gamma + 4\gamma_{\mu\nu}\star\partial_\lambda\gamma^{\mu\kappa}\star\partial_\kappa\gamma^{\nu\lambda})], \tag{15}$$

$$S^{(2)}_{\rm grav} = \int d^4x[\frac{1}{4}(4\gamma_{\kappa\alpha}\star\gamma^{\alpha\nu}\star\partial^\rho\gamma^{\lambda\kappa}\star\partial_\rho\gamma_{\nu\lambda} + (2\gamma_{\lambda\mu}\star\gamma_{\kappa\nu} - \gamma_{\mu\kappa}\star\gamma_{\nu\lambda})\star\partial^\rho\gamma^{\mu\kappa}\star\partial_\rho\gamma^{\nu\lambda}$$
$$-2\gamma_{\lambda\alpha}\star\gamma_\nu{}^\alpha\star\partial^\rho\gamma^{\lambda\nu}\star\partial_\rho\gamma - 2\gamma^{\rho\sigma}\star\gamma_\nu{}^\kappa\star\partial_\rho\gamma_{\lambda\kappa}\star\partial_\sigma\gamma^{\nu\lambda}$$
$$+\gamma^{\rho\sigma}\star\gamma^{\nu\lambda}\star\partial_\sigma\gamma_{\nu\lambda}\star\partial_\rho\gamma - 2\gamma_{\mu\alpha}\star\gamma^{\alpha\nu}\star\partial^\lambda\gamma^{\mu\kappa}\star\partial_\kappa\gamma_{\nu\lambda})], \tag{16}$$

where $\gamma = \gamma^\alpha{}_\alpha$. We have taken into account that the kinetic term in the action $S^{(0)}_{\rm grav}$ involves quadratic $\star$-product terms under the integral sign and is the same as the commutative case. Thus, for example,

$$\int d^4x\partial_\sigma\gamma_{\lambda\rho}\star\partial^\sigma\gamma^{\lambda\rho}$$
$$= \int d^4x\partial_\sigma\gamma_{\lambda\rho}\partial^\sigma\gamma^{\lambda\rho}. \tag{17}$$

It follows from this that the two-point graviton propagator is the same as in the commutative theory. The graviton propagator in the fixed de Donder gauge $\alpha = -1$ [29] is given by

$$D^{\rm grav}_{\mu\nu\rho\sigma}(x) = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})$$
$$\times\left(\frac{-i}{(2\pi)^4}\right)\int\frac{d^4p}{p^2 - i\epsilon}\exp[ip\cdot(x - x')]. \tag{18}$$

In momentum space this becomes

$$D^{\rm grav}_{\mu\nu\rho\sigma}(p) = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})\frac{1}{p^2}. \tag{19}$$

The ghost propagator in momentum space is given by

$$D^G_{\mu\nu}(p) = \frac{\eta_{\mu\nu}}{p^2}. \tag{20}$$

Let us consider the effects of an infinitesimal gauge transformation

$$x^{'\mu} = x^\mu + \zeta^\mu \tag{21}$$

on the noncommutative generating functional $Z[j_{\mu\nu}]$, where ζ^μ can depend on x^μ and $\gamma^{\mu\nu}$. We get

$$\delta\mathbf{g}^{\mu\nu}(x) = -\zeta^\lambda(x)\star\partial_\lambda\mathbf{g}^{\mu\nu}(x) + \partial_\rho\zeta^\mu(x)\star\mathbf{g}^{\rho\nu}(x)$$
$$+\partial_\sigma\zeta^\nu(x)\star\mathbf{g}^{\mu\sigma}(x) - \partial_\alpha\zeta^\alpha(x)\star\mathbf{g}^{\mu\nu}(x). \tag{22}$$

We now find that

$$\delta\gamma_{\mu\nu}(x) = -\zeta^\lambda\star\partial_\lambda\gamma_{\mu\nu} + \partial^\rho\zeta_\mu\star\gamma_{\rho\nu} + \partial^\rho\zeta_\nu\star\gamma_{\mu\rho}$$
$$-\partial^\rho\zeta_\rho\star\gamma_{\mu\nu} + \frac{1}{\kappa}(\partial_\nu\zeta_\mu + \partial_\mu\zeta_\nu - \partial^\rho\zeta_\rho\eta_{\mu\nu}). \tag{23}$$

The functional generator Z is invariant under changes in the integration variable and the transformation (23).

The noncommutative gauge transformations (23) should be considered part of an $NCSO(3,1)$ group of gauge transformations, which could correspond in general to a complex gravity theory of the kind discussed in refs. [27, 7]. Bonora et al. [28] have shown that local gauge theories in commutative spaces with $\theta = 0$ do not trivially extend themselves to noncommutative spaces. It is clear that in the limit $\theta \to 0$ the standard local Lorentz group of gauge transformations $SO(3,1)$ is recovered. By considering charge conjugation operation in gauge theories, Bonora et al. can construct a $NCSO(3,1)$ group of gauge transformations.

3 Gravitational Self-Energy

The lowest order contributions to the graviton self-energy will include the standard graviton loops, the ghost field loop contributions and the measure loop contributions. In perturbative gravity theory, the first order vacuum polarization tensor $\Pi^{\mu\nu\rho\sigma}$ must satisfy the Slavnov-Ward identities [23]:

$$p_\mu p_\rho D^{\mu\nu\alpha\beta}(p)\Pi_{\alpha\beta\gamma\delta}(p)D^{\gamma\delta\rho\sigma}(p) = 0. \tag{24}$$

By symmetry and Lorentz invariance, the vacuum polarization tensor must have the form

$$\Pi_{\alpha\beta\gamma\delta}(p) = \Pi_1(p^2)p^4\eta_{\alpha\beta}\eta_{\gamma\delta} + \Pi_2(p^2)p^4(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma})$$
$$+\Pi_3(p^2)p^2(\eta_{\alpha\beta}p_\gamma p_\delta + \eta_{\gamma\delta}p_\alpha p_\beta) + \Pi_4(p^2)p^2(\eta_{\alpha\gamma}p_\beta p_\delta + \eta_{\alpha\delta}p_\beta p_\gamma$$
$$+\eta_{\beta\gamma}p_\alpha p_\delta + \eta_{\beta\delta}p_\alpha p_\gamma) + \Pi_5(p^2)p_\alpha p_\beta p_\gamma p_\delta. \tag{25}$$

The Slavnov-Ward identities impose the restrictions

$$\Pi_2 + \Pi_4 = 0, \quad 4(\Pi_1 + \Pi_2 - \Pi_3) + \Pi_5 = 0. \tag{26}$$

For noncommutative gravity theory, it is useful to treat the metric $g_{\mu\nu}$ as an $N \times N$ matrix and to employ double line notation when calculating Feynman

graphs. The planar graphs and the non-planar graphs are not equivalent for $\theta \neq 0$. For the planar graviton graphs, we can treat the momentum as an additional index flowing along double lines [30]. For an L loop planar graph, the momenta of all lines may be expressed in terms of momenta $r_1, ...r_{L+1}$ that run along an index line of the graph. The momentum through any propagator or external line is given by $r_i - r_j$, where r_i and r_j are the index momenta that run along the sides of the propagator. This configuration automatically guarantees momentum conservation at the vertices. The index momenta along adjacent sides of a graviton propagator move in opposite directions. This construction is not valid for non-planar graphs.

In momentum space, a graviton interaction vertex has an additional phase factor relative to the commutative theory [8, 13, 14]

$$V(q_1, q_2..., q_n) = \exp\biggl(-\frac{i}{2}\sum_{i<j} q_i \times q_j\biggr), \tag{27}$$

where

$$q_i \times q_j \equiv q_{i\mu}\theta^{\mu\nu}q_{j\nu}. \tag{28}$$

In flat spacetime, this is the only change of the Feynman rules. Using momentum conservation, we find that $V(q_1, ..., q_n)$ is invariant under the cyclic permutations of q_i.

For a planar graph (genus zero), the phase factor associated with any internal propagator is equal and opposite at its two end vertices, thereby cancelling identically. Thus, the planar graviton graph has a phase factor

$$V(p_1, p_2, ..., p_n) = \exp\biggl(-\frac{i}{2}\sum_{i<j} p_i \times p_j\biggr), \tag{29}$$

where the sum is taken over all external momenta in the appropriate cylic order.

The basic lowest order graviton self-energy diagram is determined by

$$\Pi^{\rm planar}_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\kappa^2 \exp\biggl(-\frac{i}{2}\sum_{i<j} p_i \times p_j\biggr)\int d^4q\,\mathcal{U}_{\mu\nu\alpha\beta\gamma\delta}(p, -q, q-p)D^{{\rm grav}\alpha\beta\kappa\lambda}(q)$$
$$\times D^{{\rm grav}\gamma\delta\tau\xi}(p-q)\mathcal{U}_{\kappa\lambda\tau\xi\rho\sigma}(q, p-q, -p), \tag{30}$$

where $i, j = 1, 2, 3$ and $\mathcal{U}$ is the three-graviton vertex function

$$\mathcal{U}_{\mu\nu\rho\sigma\delta\tau}(q_1, q_2, q_3) = -\frac{1}{2}\biggl[q_{2(\mu}q_{3\nu)}\biggl(2\eta_{\rho(\delta}\eta_{\tau)\sigma} - \eta_{\rho\sigma}\eta_{\delta\tau}\biggr)$$
$$+q_{1(\rho}q_{3\sigma)}\biggl(2\eta_{\mu(\delta}\eta_{\tau)\nu} - \eta_{\mu\nu}\eta_{\delta\tau}\biggr) + ...\biggr], \tag{31}$$

and the ellipsis denote similar contributions. The sum in the phase factor is taken over all the external momenta p_i. We must add to this result the contributions from the planar one loop fictitious ghost particle graph and tadpole

graph, in which the noncommutative phase factor again only depends on the external momenta. The θ dependent phase factor is present in all graviton interaction terms and in all graviton tree planar graphs.

The non-planar graviton graphs have propagators that intersect one another or cross external lines. Any non-planar graph has an additional phase $\exp(iq_i \times q_j)$ for each momenta q_i and q_j that cross, in addition to the external momenta phase factor. The complete phase for a general graviton graph is

$$W(p_1, p_2, ...p_n; q_1, q_2, ...q_n) = V(p_1, p_2, ..., p_n) \exp\left[-i\left(\frac{1}{2}\sum_{i,j} I_{ij} q_i \times q_j\right)\right], \quad (32)$$

where I_{ij} is the intersection matrix that counts the number of times the ith graviton line crosses over the jth graviton line.

The non-planar one loop self-energy diagram is given by

$$\Pi^{\rm nonplanar}_{\mu\nu\rho\sigma}(p) = \frac{1}{2}\kappa^2 \exp\left(-\frac{i}{2}\sum_{i<j} p_i \times p_j\right) \int d^4q \exp(iq_i \times p_j) \mathcal{U}_{\mu\nu\alpha\beta\gamma\delta}(p, -q, q-p)$$

$$D^{{\rm grav}\alpha\beta\kappa\lambda}(q) D^{{\rm grav}\gamma\delta\tau\xi}(p-q) \mathcal{U}_{\kappa\lambda\tau\xi\rho\sigma}(q, p-q, -p). \quad (33)$$

To this diagram, we must add the non-planar fictitious ghost particle graph and tadpole graph contributions, which also have an additional internal momentum phase factor. The final graviton self-energy one loop, non-planar result for the noncommutative theory will be convergent for values of the external momenta, because of the rapid oscillations of the phase factor $\exp(iq \times p)$, where q is an internal momentum and p is an external momentum. However, the non-planar graph is singular when $|p_\mu \theta^{\mu\nu}|$ vanishes, because the phase factor is then zero. The effective gravitational cutoff in momentum space is

$$\Lambda_{\rm grav} = \frac{1}{\sqrt{-p_\mu \theta^2_{\mu\nu} p_\nu}}. \quad (34)$$

Therefore, as in scalar field theories [14], the perturbative quantum gravity theory has a peculiar mixture of ultraviolet and infrared divergent behavior. Switching on θ can replace the ultraviolet divergence with a singular infrared behavior.

We see that the divergence of the planar one loop self-energy graph is essentially the same as for the commutative perturbative result. This basic result will hold for higher order loops, as well. Therefore, we conclude that, except for the first order loop graph in matter-free gravity, noncommutative perturbative quantum gravity is unrenormalizable. However, the situation with noncommutative renormalizability of field theories that are renormalizable in the $\theta = 0$ commutative limit is not straightforward [13, 14]. For scalar noncommutative quantum field theory, the class of renormalizable, topologically nontrivial diagrams is smaller than the class of all diagrams in the theory, which leaves open the question of perturbative renormalizability of such noncommutative field theories.

In the framework of an effective gravitational field theory [26], the leading lowest order loop divergence can be "renormalized" by being absorbed into two parameters c_1 and c_2. For a non-flat spacetime background metric $\bar{g}_{\mu\nu}$, the divergent noncommutative term at one loop due to graviton and ghost loops is given in the Lagrangian by [4]:

$$\mathcal{L}^{\rm div}_{\rm 1loop} = \frac{1}{8\pi^2\epsilon}\left[\frac{1}{120}\bar{R}\star\bar{R} + \frac{7}{20}\bar{R}_{\mu\nu}\star\bar{R}^{\mu\nu}\right], \tag{35}$$

where in the dimensional regularization method $\epsilon = 4 - d$, and the effective field theory renormalization parameters are

$$c_1^{(r)} = c_1 + \frac{1}{960\pi^2\epsilon}, \quad c_2^{(r)} = c_2 + \frac{7}{160\pi^2\epsilon}. \tag{36}$$

4 Nonlocality and Unitarity

The infinite derivatives that occur in the Moyal $\star$-product render the noncommutative field theories nonlocal. This is certainly true for noncommutative quantum gravity. If we write the star product of two first order $\gamma_{\mu\nu}$ in position space as [14]:

$$(\gamma_{1\mu\nu}\star\gamma_{2\sigma\rho})(z) = \int d^4z_1 d^4z_2 \gamma_{1\mu\nu}(z_1)\gamma_{2\sigma\rho}(z_2)K(z_1,z_2,z), \tag{37}$$

where the kernel K is given by

$$K(z_1,z_2,z) = \frac{1}{\det\theta}\exp[2i(z-z_1)^\mu\theta^{-1}_{\mu\nu}(z-z_2)^\nu], \tag{38}$$

then in view of the fact that $|K|$ is a constant independent of z_1, z_2 and z, the $\star$-product appears to be infinitely nonlocal, although the oscillations in the phase of K damp out parts of the integration. If $\gamma_{\mu\nu}$ is nonzero over a tiny region of space of size $\Delta \ll \sqrt{\theta}$, then $\gamma\star\gamma$ is non-vanishing over a much larger region of size θ/Δ.

This nonlocal behaviour of the gravitational interactions has serious consequences for the dynamics. In particular, if $\theta^{i0} \neq 0$, then there is a nonlocality in time which can violate unitarity of the graviton scattering amplitudes [31] and disqualify the use of standard quantum field theory techniques, such as canonical Hamiltonian quantization. These problems surface in the investigation of open string theory calculations of scattering amplitudes [32, 33]. It is far from obvious why nature should choose to retain only the $\theta^{ij} \neq 0$ structure of the noncommutative theory, and not include the more problematical $\theta^{0i} \neq 0$ structure. Such a circumstance would render the whole scheme non-covariant and mathematically, if not physically, displeasing.

One important problem that can arise, because of the nonlocal nature of the noncommutative field theories is that the theories can exhibit instability problems. This has always been an open question in string theories and in D-brane physics [34, 35, 3]. It is certainly also an open question in our perturbative noncommutative quantum gravity.

5 Conclusions

We have developed a perturbative, noncommutative quantum gravity formalism by using the Moyal $\star$-product in the gravity action, wherever products of gravitational fields and their derivatives occur. By expanding about Minkowski flat spacetime, we were able to calculate the planar and non-planar loop graphs to first order. We find that by using the Feynman rules appropriate for noncommutative quantum field theory, the planar loop graph was essentially the same as in commutative perturbative quantum gravity, up to a non-zero phase factor that only depends on the external momenta. Only the non-planar loops exhibit convergence properties, because of the non-vanishing phase factors that depend on the internal loop momenta, and this was due to their generically oscillatory behavior. The dependence on the cut-off $\Lambda_{\rm grav}$ was such that there is a peculiar mixture of ultraviolet and infrared singular behavior.

The main disappointing result of our investigation is that perturbative noncommutative quantum gravity is not renormalizable, due to the divergent behavior of all the planar loop graphs to all orders, giving rise to new coefficients in the graviton self-energy contributions at every order and, therefore, yielding an infinite number of parameters to all orders. This result should perhaps come as a surprise, because the assumption that the spacetime coordinates are noncommuting is quite a drastic one; it implies the existence of an essentially lattice structure of spacetime. Due to the "fuzzy" nature of spacetime that arises from the Heizenberg uncertainty principle

$$\Delta x^\mu \Delta x^\nu \geq |\theta^{\mu\nu}|, \tag{39}$$

and the intrinsic nonlocality of the interactions, we might expect that the quantum gravity calculations would be finite to all orders. Of course, these results are based on perturbative calculations, which do not provide an answer to whether a non-perturbative, noncommutative quantum gravity formalism could produce a finite, unitary and gauge invariant graviton scattering amplitude. We are far from being able answer this question in the affirmative.

In ref. [11], a complex symmetric metric formulation of quantum gravity was investigated, in which a Moyal $\diamond$-product of functions was introduced for *anticommuting* coordinates. This product was defined in terms of a symmetric two-tensor $\tau^{\mu\nu} = \tau^{\nu\mu}$ and could lead to possible convergence of planar loop graphs.

The nonlocal nature of the noncommutative gravitational interactions, raises some serious questions about the physical stability of such a theory and whether the whole scheme leads to physically acceptable consequences. Clearly, further investigation of these questions is needed before we can accept that a noncommutative field theory program, and a noncommutative quantum gravity theory, can lead to physically viable theories of nature. All of these issues do have important ramifications for string theory and recent developments in D-brane physics.

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